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R WAVE PROPAGATION NEAR THE CUT-OFF FREQUENCY

S.S. Sazhin
Fluent Europe Ltd., Hutton's Buildings, 146 West Street, Sheffield, S1 4ES, U.K.

N.M. Temme
Centrum voor Wiskunde en Informatica, Kruislaan 413,
1098 SJ, Amsterdam, The Netherlands

ABSTRACT

R wave propagation at frequencies near the cut-off frequency is studied based on a weakly relativistic approximation. It is pointed out that the deviation of the relativistic cut-off frequency from the corresponding value of the cut-off frequency predicted by the theory of wave propagation in a cold plasma is particularly noticeable in a rarefied plasma. The obtained results are applicable to the analysis of low frequency cut-off in the dynamic spectra of natural magnetospheric radio emissions at frequencies above the electron plasma frequency (nonthermal continuum).

I. INTRODUCTION

It is known that, for certain values of wave frequencies and plasma parameters, the dispersion equation for wave propagation in a cold plasma has solutions corresponding to the wave refractive index N equal to zero [1]. Frequencies at which this happens are called cut-off frequencies. The analysis of wave propagation in the vicinity of these frequencies is particularly important for understanding different properties of wave propagation in plasmas, and, in particular, in the Earth's magnetosphere, and also for wave diagnostics of plasma parameters [2 – 4]. Although the results of the latter diagnostics based on a cold plasma theory appeared to be in a qualitative agreement with the *in situ* measurements of these parameters, the more refined quantitative diagnostics requires taking into account the effects of finite plasma temperature as well. One can easily see that the finite temperature effects do not inluence wave propagation within the nonrelativistic theory, but this influence appears, and can be quite significant, if we base our analysis on a more general weakly relativistic approximation. Some basic results of an approximate relativistic theory of plasma cut-offs were earlier

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discussed in [5], where it was pointed out that the relativistic effects can lead to a noticeable shift in the cut-off frequencies. In view of the importance of this shift for practical applications (in particular, for diagnostics of plasma parameters in the Earth's magnetosphere), we feel that further developments of the relativistic theory of plasma cut-offs would be justified.

The results of one of these developments have been presented by Sazhin and Temme [6] and we will describe them briefly. In contrast to [5] we restrict ourselves to the analysis of only one type of wave, namely the so called R wave propagating strictly parallel to the magnetic field. In the cold plasma limit this wave is described by the following dispersion equation [1]:

$$N^2 = R, \tag{1}$$

where

$$R = 1 + \frac{\nu Y^2}{Y-1}, \tag{2}$$

$\nu = \Pi^2/\Omega^2$, $Y = \Omega/\omega \leq (\sqrt{1+4\nu}-1)/2\nu \equiv \Omega/\omega_{cf0}$, Π is an electron plasma frequency, Ω is the electron gyrofrequency, ω is the wave frequency.

In a weakly relativistic approximation the dispersion equation for this wave can be written as:

$$N^2 = 1 - \frac{2\nu Y^2}{r}\left[\mathcal{F}_{1/2,2} - \frac{d\mathcal{F}_{3/2,2}}{dz}(A_e - 1)N^2\right], \tag{3}$$

where:

$$\mathcal{F}_{q,p} \equiv \mathcal{F}_{q,p}(z,a,b) = -i\int_0^\infty e^{izt-\frac{at^2}{1-it}}(1-it)^{-q}(1-ibt)^{-p}dt, \tag{4}$$

is the generalized Shkarofsky function, $z = 2(1-Y)/r$, $a = N^2/r$, $r = p_{0\|}^2/(m_e^2c^2)$, $b \equiv A_e = p_{0\perp}^2/p_{0\|}^2$. When deriving (3) we assumed that the electron distribution function had the the form:

$$f(p_\perp, p_\|) = \left(\pi^{3/2}p_{0\perp}^2 p_{0\|}\right)^{-1}\exp\left(-\frac{p_\perp^2}{p_{0\perp}^2} - \frac{p_\|^2}{p_{0\|}^2}\right), \tag{5}$$

where $p_{0\perp(\|)}$ is the electron thermal momentum in the direction perpendicular (parallel) to the magnetic field, $p_\perp$ and $p_\|$ are the electron momenta in the corresponding directions.

II. PROPAGATION IN AN ISOTROPIC PLASMA

In the case of an isotropic plasma ($A_e = 1$) equation (3) can be simplified to:

$$N^2 = 1 - \frac{2\nu Y^2}{r}\mathcal{F}_{5/2}, \tag{6}$$

where

$$\mathcal{F}_{5/2} \equiv \mathcal{F}_{5/2}(z) = -i\int_0^\infty e^{izt - \frac{at^2}{1-it}}(1-it)^{-5/2}dt = \mathcal{F}_{1/2,2}(z, a, b=1) \tag{7}$$

is the Shkarofsky function with the index $\tilde{q} = 5/2$.

Restricting our analysis to waves at frequencies near the cutoff frequency (ω_{cf}) we can assume that $N^2 \ll 1$ and $a \ll 1$. Assuming also that $| z - a | \to \infty$ and $| \arg z | < 3\pi/2$ (which is justified for the waves under consideration) we obtain the final equation describing the propagation of the R wave at frequencies close to the cut-off frequency, ω_{cf}, in an isotropic plasma:

$$N^2 = 1 - \frac{2\nu Y^2}{r}e^{-a}\sum_{j=0}^{\infty}\frac{a^j}{j!}\sum_{\tilde{j}=0}^{\infty}\frac{(-1)^{\tilde{j}}(z-a)^{-1-\tilde{j}}\Gamma(\frac{5}{2}+j+\tilde{j})}{\Gamma(\frac{5}{2}+j)}, \tag{8}$$

where $\Gamma(z)$ is the Euler gamma function.

Keeping terms up to the second order with respect to a and z^{-1} we rewrite (8) as:

$$N^2 = R + \nu Y^2\left[\frac{5r}{4(1-Y)^2} - \frac{35r^2}{16(1-Y)^3}\right]. \tag{9}$$

At the frequency corresponding to the cut-off in a cold plasma, ω_{cf0}, when $R = 0$, and restricting ourselves to considering the contribution of terms up to the first order, we have:

$$N^2(\omega = \omega_{cf0}) = \frac{5\nu Y^2 r}{4(1-Y)^2}. \tag{10}$$

This equation is compatible with equation (3.8) given in [5] and taken in the limit $A_e = 1$. The main restriction of the equations derived in this section is that they are valid for an isotropic plasma only. In the next section we generalize the results to the case of an anisotropic plasma.

III. PROPAGATION IN AN ANISOTROPIC PLASMA

In the case of wave propagation in an anisotropic plasma we should base our analysis on the general equation (3).

For small a we can write the following expansion for $\mathcal{F}_{q,p}$ [7]:

$$\mathcal{F}_{q,p} = e^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} \phi_{j+q,p} \tag{11}$$

with

$$\phi_{q,p} = \int_0^\infty \frac{e^{-(z-a)s}}{(1+s)^q (1+bs)^p} \, ds. \tag{12}$$

The asymptotic expansion of (12) for large $z-a$ can be obtained either by expanding the denominator of the integrand for small s or by integrating (12) by parts. In both cases we obtain the following first three terms of this expansion:

$$\phi_{q,p} \sim \frac{1}{z-a} - \frac{q+bp}{(z-a)^2} + \frac{q(q+1)+2bpq+b^2p(p+1)}{(z-a)^3} + \ldots . \tag{13}$$

Having substituted (13) into (11) and keeping only terms up to the third order in a or z^{-1} we can simplify (11) to:

$$\mathcal{F}_{q,p} \sim \frac{1}{z} - \frac{q+bp}{z^2} + \frac{q(q+1)+2bpq+b^2p(p+1)}{z^3} + \ldots . \tag{14}$$

Having substituted (14) into (3) and neglecting the higher order terms, we obtain:

$$N^2 = 1 - \frac{\nu Y^2}{1-Y}\left[1 - \frac{1+4A_e}{2z} + \frac{3+8A_e+24A_e^2}{4z^2} + \frac{(A_e-1)N^2}{z}\right]. \tag{15}$$

Equation (15) can be explicitly resolved with respect to N^2. Assuming $|R|$ small and neglecting the higher order terms we obtain:

$$N^2 = R + \frac{\nu Y^2 r(1+4A_e)}{4(1-Y)^2} - \frac{R(A_e-1)\nu Y^2 r}{2(1-Y)^2}$$

$$- \frac{r^2 \nu Y^2}{16(1-Y)^3}\left[(3+8A_e+24A_e) + 2\nu Y^2(-1-3A_e+4A_e^2)(1-Y)^{-1}\right]. \tag{16}$$

Neglecting the second order terms we can further simplify (16) to:

$$N^2 = R + \frac{\nu Y^2 r(1+4A_e)}{4(1-Y)^2}. \tag{17}$$

At the cut-off frequency in a cold plasma, when $R = 0$, equation (17) gives an expression for N^2 at $\omega = \omega_{cf0}$ which exactly coincides with that derived in [5]. For isotropic plasma, expression (16) reduces to (9).

Note that equations (3.8) and (3.11) in [5] can be considered as particular cases of equation (17) corresponding to $R = 0$ and $N^2 = 0$ respectively. Equation (16) can be used either for more refined analysis of the R wave propagation in the vicinity of the cut-off frequency, or for the rigorous justification of the range of applicability of equation (17). As one can see from equation (16) the neglect of second order terms with respect to r is not always justified, particularly for Y close to 1 and/or large A_e.

In the Earth's magnetosphere the R wave cut-off was observed as a lower frequency cut-off in the dynamic spectrum of the natural electromagnetic radiation at frequencies above the electron plasma frequency (nonthermal continuum) [2–4]. Based on the theory of wave propagation in a cold plasma this cut-off was used to estimate the magnetospheric magnetic field $|\mathbf{B}_0|$ bringing results in a qualitative agreement with the *in situ* measurements of this field. However, we strongly believe that before this method of diagnostics of $|\mathbf{B}_0|$ or electron density could be recommended for practical applications, the relativistic effects on R wave propagation near the cut-off frequency need to be taken into account.

IV. CONCLUSIONS

An expression for the R wave refractive index $\mathbf{N}$ at frequencies near the cut-off frequency, where $|\mathbf{N}|$ is small, is presented in the form of a Taylor series with respect to N^2 and an asymptotic series with respect to the squared electron thermal velocity. This expression allowed us to obtain a rather accurate expression for the cut-off frequency, generalizing the expression obtained earlier [5]. It is pointed out that the relativistic corrections to the corresponding expression for the R wave cut-off frequency in a cold plasma are particularly important in case of wave propagation in a rarefied plasma.

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